INTRODUCTION TO LOGICAL LANGUAGES

LANGUAGE, LOGIC, AND COMPUTATION

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Recap

Propositional logic

Modal logic

Recap

LOGIC: RECAP

- Formal language: object language vs. meta language
- Inference system: rules and axioms
- Models: abstraction of the world.
- Truth conditional semantics: when a formula is true
- Soundness and completeness: connecting provability and validity

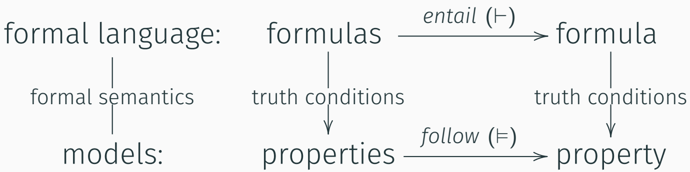
LOGIC: RECAP

- Formal language: $\texttt{All}(X, Y)$ where $X, Y \in N$
- Inference system and syntactic consequence ($\Gamma \vdash \varphi$)

$$\frac{\texttt{All}(X, Y) \qquad \texttt{All}(Y, Z)}{\texttt{All}(X, Z)}\ \text{Trans} \qquad \qquad \texttt{All}(X, X)\quad \text{Id}$$

- Truth conditional semantics
  - Model: $\mathcal{M} = \langle O, I \rangle$
  - Semantics: $\mathcal{M} \vDash \texttt{All}(X, Y) \iff I(X) \subseteq I(Y)$
  - Semantic consequence: $\Gamma \vDash \varphi$ (on all the models, if all the formulas in Γ are true then φ is also true).
- Soundness and completeness
  - Soundness: for all φ and Γ: $\Gamma \vdash \varphi$ implies $\Gamma \vDash \varphi$.
  - Completeness: for all φ and Γ: $\Gamma \vDash \varphi$ implies $\Gamma \vdash \varphi$.

THE GENERAL PICTURE

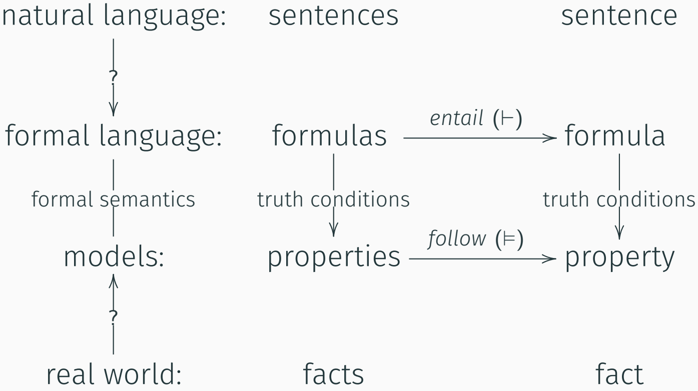


(Inspired by a similar picture from a course by Raffaella Bernardi.)

THE GENERAL PICTURE

$$
\begin{array}{cccc}
\text{formal language:} & \{\texttt{All}(A,B), \texttt{All}(B,C)\} & \xrightarrow{\textit{entail}\ (\vdash)} & \texttt{All}(A,C) \\
\Big| \text{ formal semantics} & \Big\downarrow \text{ truth conditions} & & \Big\downarrow \text{ truth conditions} \\
\text{models:} & I(A) \subseteq I(B), I(B) \subseteq I(C) & \xrightarrow{\textit{follow}\ (\vDash)} & I(A) \subseteq I(C)
\end{array}
$$

THE GENERAL PICTURE



Propositional logic

PROPOSITIONAL LOGIC (命题逻辑): SYNTAX

Given a set of proposition letters PV, we can define the propositional language by the following context-free grammar:

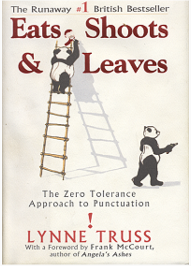
- $\varphi \longmapsto p$, where $p \in \mathsf{PV}$
- $\varphi \longmapsto \neg\varphi$,
- $\varphi \longmapsto (\varphi \wedge \varphi)$,
- $\varphi \longmapsto (\varphi \vee \varphi)$,
- $\varphi \longmapsto (\varphi \to \varphi)$.

合取 conjunction ($\wedge$): “且”, 析取 disjunction ($\vee$): “或者”, 否定 negation ($\neg$): “并非”
(实质) 蕴含 material implication ($\to$): one kind of “If … then …”.

WHY BRACKETS?

To disambiguate different readings. E.g., $p \wedge q \vee r$ may have two readings: $((p \wedge q) \vee r)$ vs. $(p \wedge (q \vee r))$. You can omit some brackets by defining the associativity power: $\neg > \wedge, \vee > \to$. E.g., $\neg p \wedge q$ is $(\neg p \wedge q)$ not $\neg(p \wedge q)$.

What animal eats, shoots and leaves?



PROPOSITIONAL LOGIC (命题逻辑): SYNTAX

Usually we use the Backus-Naur Form (BNF) to define the logical languages. The propositional language L_{PL} is defined as:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \to \varphi)$$

where $p \in \mathbf{PV}$.

It intuitively says that (equivalent to the previous CFG):

- Each basic proposition letter p is a formula
- If φ is a formula, then $\neg\varphi$ is also a formula
- If φ and ψ are formulas, then $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$ and $(\varphi \to \psi)$ are also formulas
- Only the strings constructed by the above rules are formulas.

WHAT ABOUT THE SEMANTICS?

0 for false, 1 for true:

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

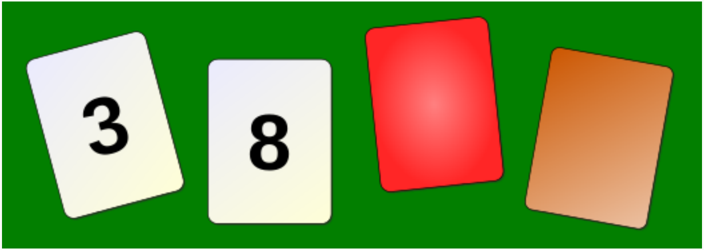
p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

p	$\neg p$
1	0
0	1

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

WASON TEST

Four cards are placed on a table, each of which has a number on one side and a colored patch on the other side. Someone claims that for each card, if one side is an even number then the other side must be red. Which card(s) must you turn over in order to test it?



SEMANTICS: MEANING OF THE WHOLE IS DETERMINED BY ITS PARTS

Models are valuations: $V : \mathbf{PV} \to \{0,1\}$, E.g.,

p	q	r	...
1	0	0	...

(functions from **PV** to $\{0,1\}$).

$$\begin{array}{lll} V \vDash p & \Leftrightarrow & V(p) = 1 \\ V \vDash \neg\varphi & \Leftrightarrow & V \nvDash \varphi \\ V \vDash \varphi \vee \psi & \Leftrightarrow & V \vDash \varphi \text{ or } V \vDash \psi \\ V \vDash \varphi \wedge \psi & \Leftrightarrow & V \vDash \varphi \text{ and } V \vDash \psi \\ V \vDash \varphi \to \psi & \Leftrightarrow & \text{If } V \vDash \varphi \text{ then } V \vDash \psi \end{array}$$

$\wedge$ and $\to$ can be actually defined by $\vee$ and $\neg$:

$$\begin{array}{lll} V \vDash \varphi \wedge \psi & \Leftrightarrow & V \vDash \neg(\neg\varphi \vee \neg\psi)) \\ V \vDash \varphi \to \psi & \Leftrightarrow & V \vDash (\neg\varphi \vee \psi) \end{array}$$

NATURAL DEDUCTION SYSTEM FOR PROPOSITIONAL LOGIC

I-$\wedge$	From p and q, infer $(p \wedge q)$.
E-$\wedge$	From $(p \wedge q)$, infer p.
	From $(p \wedge q)$, infer q.
I-$\vee$	From p, infer $(p \vee q)$.
	From q, infer $(p \vee q)$.
E-$\vee$	From $(p \vee q)$ and $(p \to r)$ and $(q \to r)$, infer r.
E-$\to$	From p and $(p \to q)$, infer q i.e. *Modus ponens* (分离规则)
I-$\to$	From [accepting p allows a proof of q], infer $(p \to q)$.
I-$\neg$	From $(p \to q)$ and $(p \to \neg q)$, infer $\neg p$.
E-$\neg$	From $\neg p$, infer $(p \to r)$.
E-$\neg\neg$	From $\neg\neg p$, infer p.

Sound and complete !

A HILBERT PROOF SYSTEM FOR PROPOSITIONAL LOGIC

Modus ponens is the only rule but there are many axioms:

$\varphi \wedge \chi \to \varphi$	E-$\wedge$
$\varphi \wedge \chi \to \chi$	
$\varphi \to (\chi \to (\varphi \wedge \chi))$	I-$\wedge$
$\varphi \to \varphi \vee \chi$	I-$\vee$
$\chi \to \varphi \vee \chi$	
$(\varphi \to \psi) \to ((\chi \to \psi) \to (\varphi \vee \chi \to \psi))$	E-$\vee$
$(\varphi \to \chi) \to ((\varphi \to \neg\chi) \to \neg\varphi)$	I-$\neg$
$\varphi \to (\neg\varphi \to \chi)$	E-$\neg$
$\varphi \vee \neg\varphi$	Excluded middle
$\varphi \to (\chi \to \varphi)$	I-$\to$
$(\varphi \to (\chi \to \psi)) \to ((\varphi \to \chi) \to (\varphi \to \psi))$	Distribute over $\to$

ANOTHER HILBERT PROOF SYSTEM FOR PROPOSITIONAL LOGIC

Consider the simpler but equivalent language of

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \to \varphi)$$

Axioms:

A1 $\varphi \to (\psi \to \varphi)$

A2 $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

A3 $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$

MP: from $\varphi \to \psi$ and φ infer ψ
Note that $(p \vee q)$ can be defined as $(\neg p \to q)$.

A SAMPLE PROOF

$$\vdash \varphi \to \varphi$$

1. $\varphi \to ((\varphi \to \varphi) \to \varphi)$ A1
2. $(\varphi \to ((\varphi \to \varphi) \to \varphi)) \to ((\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi))$ A2
3. $(\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)$ MP(1, 2)
4. $\varphi \to (\varphi \to \varphi)$ A1
5. $\varphi \to \varphi$ MP (3, 4)

MATERIAL IMPLICATION CANNOT HANDLE THOSE CONDITIONALS:

- causal conditional
- counterfactuals
- relevance implication

PARADOXES FOR MATERIAL IMPLICATION

$p \to q := \neg p \vee q$, is equivalent to $\neg(p \wedge \neg q)$. Paradoxes for material implication (valid formulas):

- $p \to (q \to p)$
- $(p \to q) \vee (q \to r)$
- $\neg(p \to q) \to (p \wedge \neg q)$
- $(p \wedge \neg p) \to q, p \to (q \vee \neg q)$

If Italy is a part of France then Rome is in France (true?) vs. If Italy is a part of France then Beijing is in France (false?). It seems that the truth of implication is not determied by the truth of antecedent and the consequent.

THE ORIGIN OF MODAL LOGIC

C. I. Lewis: strict implication ($p \strictif q$): **It is not possible that** p is true and q is false. Equivalently: **it is necessary that** if p then q. This is the origin of *modal logic*.

Modal logic

BASIC MODAL LOGIC (模态逻辑) LANGUAGE

$$\varphi \quad ::= \quad p \mid \neg\varphi \mid (\varphi \to \varphi) \mid \Box\varphi$$

where $p \in \mathbf{P}$. $\Diamond$ is usually defined as $\neg\Box\neg$. In different contexts, $\Box\varphi$ can be read differently:

- It will always be φ (in temporal logic, $G\varphi$)
- You ought to do φ (in deontic logic, $O\varphi$)
- I know that φ (in epistemic logic, $K_i\varphi$)
- After the execution of the program π, φ holds (in dynamic logic, $[\pi]\varphi$)

The truth value of $\Box\varphi$ is not fully determined by the truth value of φ.

POSSIBLE-WORLD SEMANTICS (可能世界语义)

A possible-world (or Kripke) Model $\mathcal{M} = \langle W, R, V \rangle$:

- W non-empty set.
- R is a binary relation over W (wRv means v is a possible alternative of w).
- $V : W \to \mathcal{P}(\mathbf{PV})$ gives a valuation on each world.

The semantics is defined on *pointed models* $\mathcal{M}, w$:

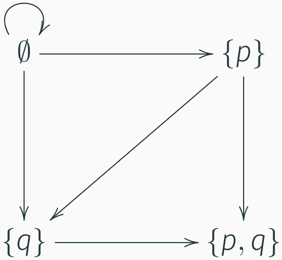
$\mathcal{M}, w \vDash p$	$\Leftrightarrow$	$p \in V(w)$
$\mathcal{M}, w \vDash \neg\varphi$	$\Leftrightarrow$	$\mathcal{M}, w \nvDash \varphi$
$\mathcal{M}, w \vDash \varphi \to \psi$	$\Leftrightarrow$	If $\mathcal{M}, w \vDash \varphi$ then $\mathcal{M}, w \vDash \psi$
$\mathcal{M}, w \vDash \Box\varphi$	$\Leftrightarrow$	for all v, if wRv then $\mathcal{M}, v \vDash \varphi$

EXAMPLES

$\mathcal{M}, w \vDash \Diamond\varphi \iff \mathcal{M}, w \vDash \neg\Box\neg\varphi \iff$ there is a v : wRv and $\mathcal{M}, v \vDash \varphi$

$\mathcal{M}, w \vDash \Box\varphi \iff \mathcal{M}, w \vDash \neg\Diamond\neg\varphi$

Where is $\Box\Diamond p$ true? Where is $\Box\Diamond p \wedge \Diamond\Box p$ true?



A HILBERT PROOF SYSTEM FOR BASIC MODAL LOGIC

System $\mathbb{K}$ consists of:

- TAUT Axioms of propositional logic
- K $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.
- NEC from $\vdash \varphi$ infer $\vdash \Box\varphi$, *rule of necessitation* (必然化规则).
- MP from $\varphi \to \psi$ and φ infer ψ.

A SAMPLE PROOF

$$\vdash_{\mathbb{K}} \Box(p \wedge q) \to \Box p \wedge \Box q$$

1	$\vdash_{\mathbb{K}} (p \wedge q) \to p$	TAUT
2	$\vdash_{\mathbb{K}} \Box((p \wedge q) \to p)$	NEC
3	$\vdash_{\mathbb{K}} \Box((p \wedge q) \to p) \to (\Box(p \wedge q) \to \Box p)$	K
4	$\vdash_{\mathbb{K}} \Box(p \wedge q) \to \Box p$	MP(4, 2)
5	$\vdash_{\mathbb{K}} \Box(p \wedge q) \to \Box q$	Repeat 1-4 for $p \wedge q \to$
6	$\vdash_{\mathbb{K}} \Box(p \wedge q) \to (\Box p \wedge \Box q)$	TAUT

SOME IMPORTANT EXTRA AXIOMS

- T $\Box p \to p$ what is necessary must be true.
- D $\Box p \to \Diamond p$ deontic logic: obligation implies permission
- 4 $\Box p \to \Box\Box p$ forever means forever forever
- 5 $\neg\Box p \to \Box\neg\Box p$ you know that you do not know
- .3 $(\Diamond p \wedge \Diamond q) \to \Diamond(p \wedge q) \vee \Diamond(p \wedge \Diamond q) \vee \Diamond(q \wedge \Diamond p)$ no branching future.