Logic: A minimal primer

Language, Logic and Computation

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Recap

A tiny formal language

A truth conditional semantics

Soundness (可靠性) 与 Completeness (完全性)

Recap

REASONING

1. Either there are colorless green ideas or there aren't any.
2. There are 24 people in the classroom.
3. All logicians are insane and some of them are not.

(1) seems to be always true no matter what a colorless green idea is, while (3) seems to be always false. Moreover, we can replace *colorless green ideas* in (1) or *logicians* in (3) by any other things the claims are still the same. Some words are special.

VALID VS. INVALID

All logicians are insane   Some linguists are logicians
___
Some linguists are insane

All X are Y Some Z are X
___
Some Z are Y

All logicians are insane   Some logicians are linguists
___
Some linguists are insane

All logicians are insane   Most logicians are linguists
___ ×
Most linguists are insane

WHY SOME INFERENCES ARE OK BUT OTHERS ARE NOT?

Logic, in a (greatly simplified) nutshell, *formally* studies the *valid* inference *patterns*.

- Analogy of a perfectly sealed water pipe: put purified water in and get purified water out.
- If the premises are true then the logical inference guarantees that the conclusions are also true.
- To define truth, logicians proposed formal semantics and use technical tools to study it.
- It provides fundamental tools for semantics of natural language. Good to know the basics of logic as a linguist.
- We will see other connections later on.

SOME NOTIONS ABOUT SETS

- Set (集合): { 马化腾, 马云 }, $\{x \mid x$ 不戴眼镜且是中国人 $\}$
- $\emptyset$: empty set.
- $x \in X$: x is in X.
- $X \subseteq Y$: X is a subset of Y, and X can be $\emptyset$.
- $X \cap Y$: the intersection of X and Y.
- $\mathcal{P}(X)$: the power set (幂集) of X, i.e. the set of all the subsets of X.

A tiny formal language

A TINY LOGICAL LANGUAGE L_{all}

Given a set of symbols $N = \{A, B, C, D, \ldots\}$, let L_{All} be the set of finite strings (*formulas*, sentences) of the following shape (where $X, Y \in N$):

- `All`(X, Y).

The *intended reading* of `All`(X, Y) is *All X are Y* where X and Y can be viewed as some nouns, e.g., animals, birds etc.

A formal (logical) language is also a collection of finite strings featuring some "logical words" with non-ambiguous structures.

Objective Language vs. Meta Language: Why do we distinguish `All`(X, Y) from *All X are Y*? Different NL sentences may have the same *Logical Form*: *Each X is Y, Every X is Y, Any X is Y...*

AXIOM AND INFERENCE RULE

Given the intended reading, the following reasoning pattern sounds 靠谱 (for any $X, Y \in N$):

$$\frac{\texttt{All}(X, Y) \qquad \texttt{All}(Y, Z)}{\texttt{All}(X, Z)}\ \text{Trans} \qquad\qquad \frac{}{\texttt{All}(X, X)}\ \text{Id}$$

We call those rules with the empty premises, the *axioms* (more precisely, axiom schemata (公理模式) if there are variables).

Trans and Id form an inference system called S_{all}.

DERIVATIONS AND PROOFS

We are interested in what sentences can be derived from what premises according to the inference system. Given the set of premises: $\Gamma = \{\texttt{All}(A, B), \texttt{All}(B, C), \texttt{All}(C, B), \texttt{All}(C, D)\}$

$$\dfrac{\texttt{All}(B, C) \qquad \texttt{All}(C, B)}{\texttt{All}(B, B)}\,(\text{Trans})$$

$$\dfrac{\dfrac{\texttt{All}(A, B) \qquad \texttt{All}(B, C)}{\texttt{All}(A, C)}\,(\text{Trans}) \qquad \texttt{All}(C, D)}{\texttt{All}(A, D)}\,(\text{Trans})$$

$\Gamma \vdash_S \varphi$ (Γ derives φ in S) *iff* there is a finite derivation tree such that the root is φ and the "leaves" are either in Γ or instances of the axioms and the branches are connected by the rules.

DERIVATIONS AND PROOFS

$$\dfrac{\dfrac{\texttt{All}(A,B) \qquad \texttt{All}(B,C)}{\texttt{All}(A,C)}\ (\text{Trans}) \qquad \texttt{All}(C,D)}{\texttt{All}(A,D)}\ (\text{Trans})$$

We can turn the tree into a linear form (like a math proof).

1. $\texttt{All}(A, B)$ (Γ)
2. $\texttt{All}(B, C)$ (Γ)
3. $\texttt{All}(A, C)$ (Trans (1)(2))
4. $\texttt{All}(C, D)$ (Γ)
5. $\texttt{All}(A, D)$ (Trans (3)(4))

$\Gamma \vdash_S \varphi$ iff there is a finite sequence of sentences in L such that each sentence in this sequence is either an instance of the axiom in S or in Γ or it is obtained by using some rule of S and the previous sentences in the sequence.

DIFFERENT INFERENCE SYSTEMS

- Mainly axioms, few rules: *Hilbert System*;
- Mainly rules, few (or no) axioms: *Natural Deduction System, Sequent Calculus*...

WHY DO WE TRUST THE AXIOM AND THE RULE?

Because:

- The axiom is always true, and
- The rule preserves the truth.
- All together, if you start from true premises, you only get true conclusions: the reasoning is valid.

But what is the concept of *truth* here?

Are the axioms and rules enough?

A truth conditional semantics

FORMAL MODEL

Whether a sentence is true or false depends on the meaning of the symbols in N and the situation in concern. We use abstract mathematical *model* to represent them.

A model for L_{All} is an *ordered pair*: $\mathcal{M} = \langle O, I \rangle$:

- O is a set, e.g., of all the relevant people in concern
- $I : N \to \mathcal{P}(O)$ is an interpretation function assigning each $X \in N$ to a subset of O.

The O can be different in different models.

SEMANTICS

The following truth condition defines which sentences are true on which models (the condition when $\mathcal{M}$ satisfies $\texttt{All}(X, Y)$):

$$\boxed{\mathcal{M} \vDash \texttt{All}(X, Y) \quad \text{iff} \quad I(X) \subseteq I(Y)}$$

We write $\Gamma \vDash \varphi$ if for *all* models $\mathcal{M}$: $\mathcal{M} \vDash \Gamma$ implies $\mathcal{M} \vDash \varphi$.

If $\Gamma \vDash \varphi$, we say φ is a *semantic consequence* of Γ (whenever Γ is all true then φ is true, i.e., it is valid to infer φ from Γ).

Now how to connect proof ($\vdash$) and validity ($\vDash$)?

Soundness (可靠性) 与 Completeness (完全性)

SOUNDNESS AND COMPLETENESS

Note that $\vdash_S$ and $\vDash$ are differently defined!

Soundness ($\vdash_S \implies \vDash$)

For any Γ and φ, if $\Gamma \vdash_S \varphi$ then $\Gamma \vDash \varphi$

Compeleteness ($\vDash \implies \vdash_S$)

For any Γ and φ, if $\Gamma \vDash \varphi$ then $\Gamma \vdash_S \varphi$

We hope:

Sound and Complete

For all Γ and φ, $\Gamma \vdash_S \varphi$ iff $\Gamma \vDash \varphi$

In particular, given soundness, $\Gamma \nvDash \varphi \implies \Gamma \nvdash_S \varphi$, we just need to find a counter example.

SOUNDNESS: FOR ANY Γ AND φ, IF $\Gamma \vdash_{S_{all}} \varphi$ THEN $\Gamma \vDash \varphi$

Suppose $\Gamma \vdash_{S_{all}} \varphi$, by induction on the length (l) of the proof:

- $l = 1$: $\varphi \in \Gamma$ or φ=$\texttt{All}(X, X)$ for some $X \in N$. It is clear: $\Gamma \vDash \varphi$.
- Induction Hypothesis (IH): it holds when $l = k$ that $\Gamma \vDash \varphi$.
- If $l = k + 1$: the last step has three cases: the above two cases and the case when the last sentence is obtained by using the rule Trans, for example:
  - ..... many steps
  - $\texttt{All}(A, B)$
  - ..... many steps
  - $\texttt{All}(B, C)$
  - ..... many steps
  - $\texttt{All}(A, C)$ (Trans)

  By IH $\Gamma \vDash \texttt{All}(A, B)$, $\texttt{All}(B, C)$, by semantics $\Gamma \vDash \texttt{All}(A, C)$.

COMPLETENESS: FOR ANY Γ AND φ, IF $\Gamma \vDash \varphi$ THEN $\Gamma \vdash_{S_{all}} \varphi$

Given Γ, we need to build a special model $\mathcal{M}^\Gamma$ by syntactic materials of Γ and S_{all} such that Γ holds in the model, and $\mathcal{M}^\Gamma \vDash \varphi$ implies $\Gamma \vdash_{S_{all}} \varphi$.

$\mathcal{M}^\Gamma = \langle O, I \rangle$:

- $O = N$ (the set of symbols)
- for each $X \in N$, let $I(X) = \{Y \mid \Gamma \vdash_{S_{all}} \texttt{All}(Y,X)\}$.

We will show:

1. $\mathcal{M}^\Gamma \vDash \Gamma$.
2. $\mathcal{M}^\Gamma \vDash \varphi$ implies $\Gamma \vdash_{S_{all}} \varphi$.

COMPLETENESS: FOR ANY Γ AND φ, IF $\Gamma \vDash \varphi$ THEN $\Gamma \vdash \varphi$

$\mathcal{M}^\Gamma = \langle N, I \rangle$:

- for each $X \in N$, let $I(X) = \{Y \mid \Gamma \vdash_{S_{all}} \texttt{All}(\mathrm{Y,X})\}$

We show that:

1. $\mathcal{M}^\Gamma \vDash \Gamma$. recall: $\texttt{All}(X, Y)$ is true iff $I(X) \subseteq I(Y)$

For all $\texttt{All}(A, B) \in \Gamma$, $I(A) = \{Y \mid \Gamma \vdash_{S_{all}} \texttt{All}(\mathrm{Y,A})\}$
$I(B) = \{Y \mid \Gamma \vdash_{S_{all}} \texttt{All}(\mathrm{Y,B})\}$. We show $I(A) \subseteq I(B)$ by Trans.

COMPLETENESS: FOR ANY Γ AND φ, IF $\Gamma \models \varphi$ THEN $\Gamma \vdash \varphi$

$\mathcal{M}^\Gamma = \langle O, I \rangle$:

- $O = N$
- For each $X \in N$, let $I(X) = \{Y \mid \Gamma \vdash \text{All } Y \text{ are } X\}$

We now show :

2. $\mathcal{M}^\Gamma \models \varphi$ implies $\Gamma \vdash_{S_{all}} \varphi$.

For each φ=$\texttt{All}(A, B)$, if $\mathcal{M}^\Gamma \models \varphi$ then by the semantics $I(A) \subseteq I(B)$, i.e., $\{Y \mid \Gamma \vdash_{S_{all}} \texttt{All}(\mathrm{Y,A})\} \subseteq \{Y \mid \Gamma \vdash_{S_{all}} \texttt{All}(\mathrm{Y,B})\}$.

By Id, $\Gamma \vdash_{S_{all}} \texttt{All}(\mathrm{A,A})$, therefore $A \in I(A)$, thus $A \in I(B)$, finally $\Gamma \vdash_{S_{all}} \texttt{All}(\mathrm{A,B})$.

COMPLETENESS: FOR ANY Γ AND φ, IF $\Gamma \vDash \varphi$ THEN $\Gamma \vdash \varphi$

Given Γ, we need to build a special model $\mathcal{M}^\Gamma$ by syntactic things such that Γ holds in the model, and $\mathcal{M}^\Gamma \vDash \varphi$ implies $\Gamma \vdash \varphi$.

$\mathcal{M}^\Gamma = \langle O, I \rangle$:

- $O = N$
- for each $X \in N$, let $I(X) = \{Y \mid \Gamma \vdash_{S_{all}} \texttt{All(Y,X)}\}$

We have shown:

1. $\mathcal{M}^\Gamma \vDash \Gamma$.
2. If $\mathcal{M}^\Gamma \vDash \varphi$ then $\Gamma \vdash_{S_{all}} \varphi$.

If $\Gamma \vDash \varphi$ holds then $\mathcal{M}^\Gamma \vDash \varphi$ by (1). Then by (2) we have $\Gamma \vdash_{S_{all}} \varphi$. QED.

NOW WE HAVE SHOWN

定理 (Soundness and Completeness)

For any Γ *and* φ*:* $\Gamma \vDash \varphi$ *iff* $\Gamma \vdash_{S_{all}} \varphi$.

By the definition of $\Gamma \vdash_{S_{all}} \varphi$, there is a finite subset Δ of Γ such that $\Delta \vdash_{S_{all}} \varphi$. By the above theorem, we have:

推论 (Compactness)

$\Gamma \vDash \varphi$ *implies there is a finite subset* Δ *of* Γ *such that* $\Delta \vDash \varphi$

We can also easily decide which formulas are valid...

ANOTHER TINY LANGUAGE l_{some}

Given a set of symbols $N = \{A, B, C, D, \dots\}$, let L_{some} the set of all sentences in the following shape

- $\mathtt{Some}(X, Y)$, where $(X, Y \in N)$:

$$\boxed{\mathcal{M} \vDash \mathtt{Some}(X, Y) \quad \text{iff} \quad I(X) \cap I(Y) \neq \emptyset}$$

$$\frac{\mathtt{Some}(A, B)}{\mathtt{Some}(B, A)}\,\text{Symm} \qquad \frac{\mathtt{Some}(A, B)}{\mathtt{Some}(A, A)}\,\text{Ex}$$

IT BECOMES MORE INTERESTING $l_{all,some}$

Given a set of symbols $N = \{A, B, C, D, \dots\}$, let $L_{all,some}$ the set of all sentences in the following shapes:

- $\texttt{All}(X, Y)$
- $\texttt{Some}(X, Y)$

We also need the interaction axiom:

$$\frac{\texttt{Some}(A, B) \qquad \texttt{All}(B, C)}{\texttt{Some}(A, C)}\ (\mathrm{Int})$$

CONCLUSIONS

- Formal logical language
- Inference system
- Formal truth conditional semantics
- The distinction of syntax and semantics
- The connection of syntax and semantics: soundness and completeness

NEXT...

What about?

- No X is Y?
- φ or ψ
- If φ then ψ and χ
- Necessarily φ
- I know that you know φ
- I know which has φ
- ...

Propositional logic, first-order logic, modal logic, first-order modal logic, inquisitive logic, Categorial Grammar, Montague Grammar.... approximating the logical core of natural language.