



# EPISTEMIC LOGIC

## III SOME PROBLEMS OF THE BASIC FRAMEWORK

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Yanjing Wang

Department of Philosophy, Peking University

Oct. 10th 2018

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## RECAP: BASIC SYSTEMS

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The Epistemic Language:

$$\phi ::= \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid K_i\phi$$

It is interpreted on Kripke models  $\mathcal{M} = (S, \{\rightarrow_i\}_{i \in I}, V)$  where  $\rightarrow_i$  has certain properties (sometimes taken as an equivalence relation).

$$\boxed{\mathcal{M}, s \models K_i\phi \iff \forall t : s \rightarrow_i t \implies \mathcal{M}, t \models \phi}$$

# S5 SYSTEM (STRONGEST EPISTEMIC LOGIC)

## System S5

Axioms

**TAUT** all the instances of tautologies

**DISTK**  $K_i(p \rightarrow q) \rightarrow (K_i p \rightarrow K_i q)$

**T**  $K_i p \rightarrow p$

**4**  $K_i p \rightarrow K_i K_i p$

**5**  $\neg K_i p \rightarrow K_i \neg K_i p$

Rules

**MP**  $\frac{\phi, \phi \rightarrow \psi}{\psi}$

**NECK**  $\frac{\phi}{K_i \phi}$

**SUB**  $\frac{\phi}{\phi[p/\psi]}$

S5 is sound and strongly complete for modal logic over frames with equivalence relations.

# kd45 SYSTEM (STRONGEST DOXASTIC LOGIC)

## System $KD45$

Axioms

**TAUT** all the instances of tautologies

**DISTK**  $B_i(p \rightarrow q) \rightarrow (B_i p \rightarrow B_i q)$

**D**  $\neg B_i \perp$  (or  $B_i p \rightarrow \neg B_i \neg p$ )

**4**  $B_i p \rightarrow B_i B_i p$

**5**  $\neg B_i p \rightarrow B_i \neg B_i p$

Rules

**MP**  $\frac{\phi, \phi \rightarrow \psi}{\psi}$

**NECK**  $\frac{\psi}{\phi}$   
 $\frac{B_i \phi}{\phi}$

**SUB**  $\frac{\phi}{\phi[p/\psi]}$

$KD45$  is sound and strongly complete for modal logic over frames that are transitive, euclidean and serial.

## (S5) KNOWLEDGE = TRUE (KD45) BELIEF?

KD45 agents thought they are S5 agents.

KD45 belief plus the following definition:

$$K_i\phi \equiv_{df} \phi \wedge B_i\phi$$

Does it give us the S5 axioms of  $K_i$ ? Negative introspection does not work!

Consider the system of KD belief and S4 knowledge plus some interaction axioms:

- $B(p \rightarrow q) \rightarrow (Bp \rightarrow Bq)$
- $\neg B\perp$
- $K(p \rightarrow q) \rightarrow (Kp \rightarrow Kq)$
- $Kp \rightarrow KKp$
- $Kp \rightarrow p$
- $Kp \rightarrow Bp$
- $Bp \rightarrow KBp$
- $\neg Bp \rightarrow K\neg Bp$
- $Bp \rightarrow BKp$  (strong belief)

$B$  is actually  $KD45$  and we can show that  $Bp \leftrightarrow \neg K\neg Kp$ . Then we have Geach theorem (.2) from D.

# INTROSPECTIONS

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On positive introspection:

- If  $\{K\phi, \neg K\neg\psi\}$  is consistent
- Then  $\{K\phi, \psi\}$  is consistent
- Substitute  $\psi$  above with  $\neg K\phi$  we will have the KK principle (Axiom 4).

On negative introspection:

- Assuming 5, we have  $B : p \rightarrow K\neg K\neg p$ .
- This rule out the possibility of false beliefs.

$$\neg K_i p \rightarrow K_i \neg K_i p$$

Lenzen (1978)'s example:

- Suppose  $B_i K_i p$  but  $\neg p$ :  $i$  falsely believes that he knows  $p$ .
- then  $\neg K_i p$  (T axiom)
- therefore  $K_i \neg K_i p$  (5 axiom)
- thus  $B_i \neg K_i p$  (knowledge implies belief)
- conclude  $B_i \perp$  (under normal axioms for  $B_i$ )

We cannot express false belief of knowledge!

The equivalent form:  $\neg K_i \neg K_i p \rightarrow K_i p$  sounds more problematic:  
if you think it is possible to know then you know it.

Lenzen prefers S4.2:  $\neg K_i \neg K_i p \rightarrow K_i \neg K_i \neg p$

## POSITIVE INTROSPECTION

Williamson (1992, 2000) on inexact knowledge (assuming axiom 4 and T) where  $p_k$  means a tree is  $k$  cm tall (simplified version):

1. assuming (for all reasonable  $k$ ):  $K(p_{k+1} \rightarrow \neg K\neg p_k)$
2. assuming  $K\neg p_0$
3. by (2) and **axiom 4**:  $KK\neg p_0$
4. by (1) and **NEC** and **DIST** we have  $KK\neg p_0 \rightarrow K\neg p_1$
5. by (3) and (4):  $K\neg p_1$
6. repeat to derive  $K\neg p_k$
7. Suppose  $p_{666}$  is true then it contradicts to  $K\neg p_{666}$  (under T axiom)

(1) is fishy: From (1) we can derive  $p_{k+1} \rightarrow \neg K\neg p_k$ . Moreover, by T axiom  $Kp_{k+1} \rightarrow p_{k+1}$ . Hence, we can infer  $Kp_{k+1} \rightarrow \neg K\neg p_k$ .

## LOGICAL OMNISCIENCE

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Problematic closure rules:

- From  $\vdash \phi$  infer  $K\phi$
- From  $\vdash \phi \rightarrow \psi$  infer  $K\phi \rightarrow K\psi$
- From  $\vdash \phi \leftrightarrow \psi$  infer  $K\phi \leftrightarrow K\psi$

'Deductive closure principle':  $K(\phi \rightarrow \psi) \wedge K\phi \rightarrow K\psi$  is a different thing (which can also be challenged though, e.g., examples by Dretske (1970)).

The problem is more about the complexity of the logical reasoning!

## Ideas of the 'solutions':

- Reinterpretation: implicit knowledge; of an ideal agent
- Syntactic approach: a set of formulas
- Impossible worlds: inconsistent alternatives
- Awareness:  $K = \text{awareness} + \text{implicit knowledge}$
- Algorithmic knowledge:  $K = \text{answer by algorithm}$
- Neighbourhood semantics: still problematic (From  $\vdash \phi \leftrightarrow \psi$  infer  $K\phi \leftrightarrow K\psi$ )
- Timed knowledge: reasoning takes time

## EXTERNAL VS. INTERNAL

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- Help to model other's reasoning
- Help us to reason by ourselves

However, can the agents 'see' the model (without the real world)?

It seems OK for S5 models, what about KD45?



For the KD45 agent the model is actually:



Is it good enough to have generated submodel based on the states which are reachable from the real world?

Now consider the following S4 model (reflexive and transitive):



If the agent can see the model (without the real world) himself, then he can reason as follows (suggested by Yang Liu):

I am not considering  $\neg p$  as the only possibility, thus I must be on a  $p$  world, then I should know  $p$ ! What is wrong?

At the  $p$  world:  $K(\neg p \rightarrow K\neg p)$  is true thus  $K\neg K\neg p \rightarrow Kp$ . Note that  $\neg K\neg p$  is also true at  $p$  but we don't have  $K\neg K\neg p$  since it is S4 and you are not allowed to use negative introspection!

By having the model, you are implicitly using negative introspection: if I don't know  $\neg p$  then I know I don't know it. The meta reasoning violates the assumption (the model is your brain).

## GROUP NOTIONS

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David Lewis' *Convention* (1969):  $p :=$  every driver must drive on the right.

What kind of knowledge is enough to let people feel safe in driving on the right? Is 'everybody knows that  $p$ ' enough ( $Ep$ )?

What about everybody knows that everybody knows that  $p$  ( $EEp$ )?

No, since agent  $i$  considers possible that agent  $j$  considers possible that agent  $i$  does not know ( $\neg K_i K_j K_i p$ ) and thus agent  $j$  may drive on the left. You may argue by induction that  $E^k p$  is not enough.

$$C\phi := E\phi \wedge EE\phi \wedge EEE\phi \dots$$

$\mathcal{M}, s \models E_G \phi \iff$	for all $t$ such that $s \xrightarrow{EG} t, \mathcal{M}, t \models \phi$
$\mathcal{M}, s \models C_G \phi \iff$	for all $t$ such that $s \xrightarrow{CG} t, \mathcal{M}, t \models \phi$

where  $\xrightarrow{EG} = \bigcup_{i \in G} \xrightarrow{i}$  and  $\xrightarrow{CG} = (\xrightarrow{EG})^*$  (the reflexive transitive closure of  $\xrightarrow{EG}$ ).

Axiomatization:

- S5 system for  $K$  plus **NEC** and **DIST** for  $C_G$  and
- Fixed-Point Axiom:  $C_G \phi \leftrightarrow (\phi \wedge E_G C_G \phi)$
- Induction Axiom:  $(\phi \wedge C_G(\phi \rightarrow E_G \phi)) \rightarrow C_G \phi$

Questions:

- Can we easily have full common knowledge? e.g., consecutive numbers.
- Do we really need the full power of common knowledge in many cases? tricky case: muddy children.

Intuition: what we know if we put all of our knowledge together.

$$\mathcal{M}, s \models D_G \phi \iff \text{for all } t \text{ such that } s \xrightarrow{DG} t, \mathcal{M}, t \models \phi$$

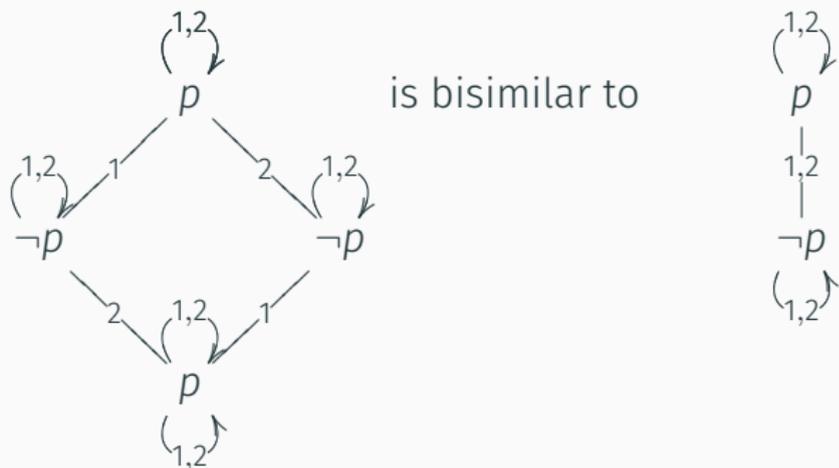
where  $\xrightarrow{DG} = \bigcap_{i \in G} \xrightarrow{i}$ .

Finite axiomatization (a wise man's knowledge):

- S5 axioms for  $D_G$  plus
- $K_i \phi \rightarrow D_G \phi$  (when  $i \in G$ )

Questions:

- It is not invariant under bisimulation;
- It is not the case:  $\{\psi \mid \mathcal{M}, s \models K_i \psi \text{ for some } i \in G\} \models \phi$  iff  $\mathcal{M}, s \models D_i \phi$  for all  $\phi$ .



- Agency
- Understanding