HYPE: A System of Hyperintensional Logic and Semantics

Hannes Leitgeb

LMU Munich

June 2019
“Hyperintensional contexts are simply contexts which do not respect logical equivalence.” (Cresswell 1975)

E.g., Barwise and Perry (1983):

- Melanie saw Jim eat an anchovy.
- Melanie saw Jim eat an anchovy and Sara eat a pickle or Sara not eat a pickle.

Compare: hyperintensional contexts generated by operators for

- seeing, belief, aprioricity, desire, aboutness, explanation, grounding,…

“The twenty-first century is seeing a hyperintensional revolution.” (Nolan 2014)

Here is a *scheme* for a semantics for (certain) hyperintensional operators:

- Let formulas be evaluated at states which may be either “world-like” or “non-world-like”.

- If one wants the *external* logic to remain classical, one can define logical consequence as strict truth preservation in “world-like” states.

- Let the semantics for modal operators be a possible-*states*-semantics:

  ![Diagram](https://via.placeholder.com/150)

  If some of these possible states are “non-world-like”, the operators will be hyperintensional, and their *internal* logic will be non-classical.

**Task 1:** Develop a state-based semantics and logic for \( \neg, \land, \lor, \rightarrow, \forall, \exists \).

**Task 2:** Develop a possible states semantics and logic for *see*, *ground*,…
In Leitgeb (2019), I introduced one way of accomplishing task 1: HYPE.

The system overlaps—but does not coincide—with:

- situation semantics (Barwise & Perry, Barwise & Etchmendy),
- data semantics (Veltman),
- truthmaker semantics (van Fraassen, Fine, Restall),
- relevance semantics (Anderson & Belnap, Routley, Dunn, Urquhart,...),
- many-valued and paraconsistent logics, impossible worlds (Belnap, Dunn, Priest,...),
- Kripke semantics for intuitionistic logic and its variations (Nelson, Gurevich, Wansing,...),
- inquisitive semantics (Ciardelli & Roelofsen).
Goals for today:

- to give a survey of HYPE, and
- to accomplish task 2—to extend HYPE to a possible states semantics for hyperintensional sentential operators.

Plan:

1. The Semantics of HYPE
2. The Logic of HYPE
3. HYPE and Semantic Paradoxes
4. Possible States Semantics for HYPE
5. Possible States Models for *Because*
6. Conclusions
I will mostly restrict myself to the language of propositional logic.

Vocabulary:

- Propositional letters: $p_1, p_2, p_3, \ldots$
- Logical symbols: $\neg, \land, \lor, \rightarrow, \top$.
- Auxiliary symbols: $(, )$.

Note that $\rightarrow$ will not express the material conditional!

Let us use the following metalinguistic abbreviations:

$\overline{p_i}$: $\neg p_i$.
$\overline{\overline{p_i}}$: $p_i$.

$A \leftrightarrow B$: $(A \rightarrow B) \land (B \rightarrow A)$.
$\bot$: $\neg \top$. 
In the semantics, a formula $A$ will be evaluated at a \textit{state} $s$:

$$s \models A,$$

which may be interpreted as:

- $A$ is satisfied in $s$, and $A$'s “inexact” subject matter is $s$,
- $s$ is an “inexact” truthmaker of $A$,
- $s$ is a way of $A$ being true,
Just like for worlds in possible worlds semantics, we will have:

- States $s$ decide disjunctions ($s \models B \lor C$ iff $s \models B$ or $s \models C$).
- States $s$ are “closed” under conjunction ($s \models B \land C$ iff $s \models B$ and $s \models C$).

  (E.g., we will have: $s \models p \lor (p \land q)$ iff $s \models p$ or $s \models p \land q$ iff $s \models p$. Which will not be equivalent to $s \models p \land (q \lor \neg q)$!)

In contrast with worlds:

- States $s$ may be incomplete ($s \not\models B$ and $s \not\models \neg B$).
- States $s$ may be overdetermined ($s \models B$ and $s \models \neg B$).

Finally,

- states $s$ may be the fusion of other states ($s = s' \circ s''$),
- states $s, s'$ may be incompatible with each other ($s \perp s'$).

Now let us make all of that formally precise.
Model $\mathcal{M} : \langle S, V, \circ, \bot \rangle$, such that:

- $S$ is a non-empty set (the set of states).
- $V : S \to \wp(\{p_1, \overline{p}_1, p_2, \overline{p}_2, p_3, \overline{p}_3, \ldots \})$ (the valuation function).
- $\circ$ is a partial function from $S \times S$ to $S$ (the fusion function), such that:
  - Either $s \circ s'$ is undefined, or $s \circ s'$ is defined (and hence in $S$) in which case we require $V(s \circ s') \supseteq V(s) \cup V(s')$.
  - $s \circ s$ is defined, and $s \circ s = s$.
  - If $s \circ s'$ is defined, then $s' \circ s$ is defined, and $s \circ s' = s' \circ s$.
  - If $(s \circ s') \circ s''$ is defined, then $s \circ ((s \circ s') \circ s'')$ is defined, and $s \circ ((s \circ s') \circ s'') = (s \circ s') \circ s''$.

- $\bot$ is a binary symmetric relation on $S$ (the incompatibility relation), s.t.:
  - If there is a $v$ with $v \in V(s)$ and $\overline{v} \in V(s')$, then $s \bot s'$.
  - If $s \bot s'$ and both $s \circ s''$ and $s' \circ s'''$ are defined, then $s \circ s'' \bot s' \circ s'''$.

For every $s$ in $S$ there is a unique $s^* \in S$ (the star image of $s$), such that:

- $V(s^*) = \{\overline{v} | v \notin V(s)\}$.
- $s^{**} = s$.
- $s$ and $s^*$ are not incompatible with each other: $s \not\bot s^*$.
- $s^*$ is largest having the last property: if $s \not\bot s'$, then $s' \circ s^* = s^*$.
The following observations make it a bit easier to see what is going on in the model assumptions:

**Observation**

For all $s, s' \in S$, define $s \leq s'$ to be the case iff $s \circ s'$ is defined and $s \circ s' = s'$. Then it follows that $\leq$ is a partial order on $S$.

**Observation**

* is antitone with respect to $\leq$: for all $s, s' \in S$, if $s \leq s'$ then $s'^* \leq s^*$. 
In a diagram:

Overdetermined, not incomplete:

\[ p, \neg p, q, \neg r \]

Incomplete, not overdetermined:

\[ p, \neg p, r \]

Overdetermined, incomplete:

\[ q, \neg q, r \]

Overdetermined, not incomplete (worlds):

\[ p, q, \neg r \]

Incomplete, not overdetermined:

\[ q, \neg r \]

Not overdetermined, not incomplete (worlds):

\[ p, q, \neg r \]
Satisfaction clauses:

- $s \models v$ iff $v \in V(s)$ (where $v$ is a literal).

- $s \models \neg A$ iff for all $s'$: if $s' \models A$ then $s \perp s'$.
  
  (In words: $s \models \neg A$ iff $s$ is incompatible with every “way” of $A$ being true.)

- $s \models A \land B$ iff $s \models A$ and $s \models B$.

- $s \models A \lor B$ iff $s \models A$ or $s \models B$.

- $s \models A \rightarrow B$ iff for all $s'$: if $s' \models A$ and $s \circ s'$ is defined, then $s \circ s' \models B$.
  
  (In words: $s \models A \rightarrow B$ iff “adding” $A$ to $s$ yields $B$.)

- $s \models \top$.

The hyperintension expressed by $A$ in a model is: $\{ s \in S \mid s \models A \}$.
Lemma (Monotonicity)

If \( s \models A \) and \( s \circ s' \) is defined, then \( s \circ s' \models A \).

Lemma (Star Satisfaction)

\( s \models \neg A \) iff \( s^* \not\models A \).

Note that the last lemma is not the satisfaction clause for \( \neg \): the lemma follows from the satisfaction clause and \( S \) being “rich” enough (i.e., for every \( s: s^* \in S \)).

(This is different from those semantics for relevance logic in which satisfaction for negation is explained by means of the Routley star.)
Define:

\[ A_1, \ldots, A_n \models B \iff \]
for all models \( \mathcal{M} \), for \( s \) in \( S \), if \( s \models A_1, \ldots, s \models A_n \), then \( s \models B \).

\[ \models B \iff \text{for all models } \mathcal{M}, \text{ for } s \text{ in } S, s \models B. \]

**Observation**

- *Every logical truth contains \( \rightarrow \) or \( \top \).*
- *Logical consequence for formulas without \( \rightarrow \) and \( \top \) coincides with First-Degree-Entailment (FDE) and has the variable-sharing property.*
- *Formulas without \( \rightarrow \) have an FDE-disjunctive-normal-form.*
- *Excluded Middle, the Law of Non-Contradiction, Explosion, Disjunctive Syllogism, and (General) Contraposition are not logically valid.*
An example from ordinary maths (cf. Edmonds and Fulkerson 1970):

- States are subsets of \{\langle e, i \rangle \mid e \text{ is an edge in a graph } G, \ i \in \{0, 1\}\}.

- Propositional letters: \(\vec{vv}'\). Define \(V, \circ, \perp\) as expected.

- Formulas will then describe actions in \(G\):
  - \(\vec{vv'}\): Walking from \(v\) to \(v'\).
  - \(\neg A\): Blocking action \(A\).
  - \(A \land B\): Carrying out \(A\) and \(B\).
  - \(A \lor B\): Carrying out \(A\) or \(B\).
  - \(A \rightarrow B\): Carrying out the sub-actions of \(B\) not contained in \(A\).

\[
\begin{align*}
\langle e_1, 1 \rangle, \langle e_2, 1 \rangle, \langle e_3, 1 \rangle &\models \vec{v_1v_4},
\langle e_1, 1 \rangle, \langle e_4, 1 \rangle, \langle e_6, 1 \rangle &\models \vec{v_1v_4},
\langle e_5, 1 \rangle, \langle e_6, 1 \rangle &\models \vec{v_1v_4}.
\end{align*}
\]

\[
\begin{align*}
\langle e_1, 0 \rangle, \langle e_5, 0 \rangle &\models \neg \vec{v_1v_4},
\langle e_1, 0 \rangle, \langle e_6, 0 \rangle &\models \neg \vec{v_1v_4},
\langle e_2, 0 \rangle, \langle e_4, 0 \rangle, \langle e_5, 0 \rangle &\models \neg \vec{v_1v_4},
\langle e_2, 0 \rangle, \langle e_6, 0 \rangle &\models \neg \vec{v_1v_4},
\langle e_3, 0 \rangle, \langle e_4, 0 \rangle, \langle e_5, 0 \rangle &\models \neg \vec{v_1v_4},
\langle e_3, 0 \rangle, \langle e_6, 0 \rangle &\models \neg \vec{v_1v_4}.
\end{align*}
\]

\[
\begin{align*}
\langle e_1, 1 \rangle &\models \vec{v_2v_4} \rightarrow \vec{v_1v_4}.
\end{align*}
\]
Another example: the *Full HYPE Model(s)*.

Restricted to the propositional letters $p, q$, the state space $S$ looks like this:

\[
\begin{array}{cccccccc}
 & p & p & q & q & & & & \\
p & p & p & q & q & & & & \\
p & p & p & q & q & & & & \\
p & q & p & q & q & & & & \\
p & q & p & q & q & & & & \\
p & p & q & q & & & & & \\
[ & p & & & & & & & ]
\end{array}
\]

Define $V, \circ, \perp$ as expected. (Worldly states are in blue.) States represent *sets of elementary content*.

In the full model, *all* formulas have a disjunctive normal form (up to equivalence at all states in the full model).

E.g., for all states $s$ in $S$ one has:

\[
s \models p \rightarrow ((p \land q) \lor \neg q) \iff s \models (p \rightarrow (p \land q)) \lor (p \rightarrow \neg q) \iff s \models q \lor \neg q
\]
The Logic of HYPE

⊢ \top.

⊢ A \rightarrow A.

⊢ A \rightarrow (B \rightarrow A).

⊢ A \rightarrow (B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)).

⊢ A \land B \rightarrow A.  ⊢ A \land B \rightarrow B.

⊢ A \rightarrow A \lor B.  ⊢ B \rightarrow A \lor B.

⊢ A \rightarrow (B \rightarrow A \land B).

⊢ (A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \lor B \rightarrow C)).

⊢ A \land (B \lor C) \leftrightarrow (A \land B) \lor (A \land C).

⊢ A \lor (B \land C) \leftrightarrow (A \lor B) \land (A \lor C).

⊢ \neg \neg A \leftrightarrow A.

⊢ \neg (A \land B) \leftrightarrow \neg A \lor \neg B.

⊢ \neg (A \lor B) \leftrightarrow \neg A \land \neg B.

\dfrac{A \rightarrow B}{\neg B \rightarrow \neg A}

A, A \rightarrow B \vdash B.
Theorem

- $A_1, \ldots, A_n, B \vdash C$ iff $A_1, \ldots, A_n \vdash B \rightarrow C$.
- $\Phi \vdash A$ iff $\Phi \models A$.
- If $\vdash C[p]$, then $\vdash C[A]$.
- HYPE has the finite model property.
- The set of logical truths of HYPE is decidable.
- HYPE has the Disjunction Property: if $\models A \lor B$, then $\models A$ or $\models B$. 
This system can be extended to first-order languages:

- $D$ is a non-empty set (the global domain).
- $SoA$ is the set of all tuples
  \[\langle P, d_1, \ldots, d_n \rangle \text{ and } \langle \overline{P}, d_1, \ldots, d_n \rangle\]
  where $P$ is an $n$-ary predicate, and $d_1, \ldots, d_n \in D$. ($SoA$ is the set of states of affairs.)
- $V: S \rightarrow \wp(SoA)$ (the valuation function).

Satisfaction at states is now defined relativized to variable assignments:

- $s, \sigma \models P(t_1, \ldots, t_n)$ iff $\langle P, \sigma(t_1), \ldots, \sigma(t_n) \rangle \in V(s)$.
- $\vdots$
- $s, \sigma \models \forall x A$ iff for all $d \in D$: $s, \sigma^d_x \models A$.
- $s, \sigma \models \exists x A$ iff there is some $d \in D$, such that $s, \sigma^d_x \models A$. 
Logic: All universal closures of the previous logical axioms of HYPE, as well as all universal closures of the additional axioms of the following types:

\[ \vdash \forall x (A \rightarrow B) \rightarrow (\forall x A \rightarrow \forall x B) \]
\[ \vdash \forall x A \rightarrow A \frac{t}{x} \]
\[ \vdash \forall x (A \rightarrow B) \rightarrow (A \rightarrow \forall x B) \text{ (where } x \text{ does not occur free in } A) \]
\[ \vdash A \frac{t}{x} \rightarrow \exists x A \]
\[ \vdash \forall x (A \rightarrow B) \rightarrow (\exists x A \rightarrow B) \text{ (where } x \text{ does not occur free in } B) \]
\[ \vdash \neg \forall x A \leftrightarrow \exists x \neg A \]

From this, one can e.g. derive

- (D) \[ \vdash \forall x (A \lor B) \leftrightarrow \forall x A \lor B \text{ (where } x \text{ does not occur free in } B) \]

**Theorem (Soundness and Completeness)**

The system is sound and complete with respect to the extended semantics.
Over and above FDE, HYPE also relates nicely to Strong Kleene Logic, the Logic of Paradox, classical logic, and intuitionistic logic:

- The logic of least states in HYPE-models is SKL (if one disregards $\rightarrow$).
- The logic of greatest states in HYPE-models is LP (if one disregards $\rightarrow$).
- Classical extensional models and classical intensional models may be viewed as special cases of HYPE-models. In them, $\rightarrow$ collapses into $\supset$, and $\neg$ collapses into classical negation.
- The logic of “wordly” states in HYPE-models is CL (if one disregards $\rightarrow$).
- Every Kripke model for intuitionistic logic with constant domains may be viewed as the partial part of a special kind of HYPE-model (where one restricts $\neg$ to $\bot$-contexts, that is, $\neg\top$-contexts).
- For all formulas $A_1, \ldots, A_n, B$ in which $\neg$ only occurs in $\bot$-contexts:
  $$A_1, \ldots, A_n \vdash_{IL+D} B \iff A_1, \ldots, A_n \vdash B.$$
- If there is a least state $s_l$ and greatest state $s_g$: $s_l \models \neg A$ iff $s_g \models A \rightarrow \bot$. 
One application of HYPE is to *semantic paradoxes*:

Kripke (1975) showed how one can determine fixed point models $M$ of, e.g., Strong Kleene logic, for languages with a type-free truth predicate $Tr$ and a background theory of syntax, such that for all $A$:

$$M \models Tr(\neg A) \iff M \models A.$$ 

Field (2008) criticized Kripke’s theory for not offering a (bi-)conditional by which these equivalences could be expressed in the object language. So he added a new conditional and determined a model $M'$ to the effect that for all $A$:

$$M' \models Tr(\neg A) \leftrightarrow A.$$ 

Downside: The new conditional is not so well-behaved, and the model $M'$ is of merely instrumental value.
(We also know one *could not* have a nice conditional, a nice consequence relation, and all T-biconditionals, because of Curry’s $\chi$: $Tr(\neg \chi \downarrow) \rightarrow \bot$.)
New idea: Have a “nice” conditional but do not aim at all T-biconditionals!

**Theorem**

There is a HYPE-model for type-free truth and syntax, such that for all states $s$: For all $A$ without $\rightarrow$ (but with quantifiers ranging over the full language),

$$s \models Tr(\neg A \neg) \leftrightarrow A.$$  

**Proof:**

Build a HYPE-Model in which the complete lattice of all “Kripkean fixed points” is used as state space.
Remarks:

- We are not guaranteed all instances of the T-scheme. Is there some rationale for that?
Remarks:

- We are not guaranteed all instances of the T-scheme. Is there some rationale for that? Yes: after all, $\leftrightarrow$ is hyperintensional!

One should not expect $\text{Tr} (\backslash \backslash A \backslash \backslash)$ and $A$ to be generally intersubstitutable in hyperintensional contexts!

$s_j |\models \text{Tr} (\backslash \backslash A \backslash \backslash) \leftrightarrow A$ expresses that $\text{Tr} (\backslash \backslash A \backslash \backslash)$ and $A$ are true in, and have, the same subject matters, they have the same truthmakers, they have the same ways of being true, they express the same hyperintension.

Perhaps this could not be so generally for $\text{Tr} (\neg \neg \neg A \rightarrow B \neg \neg \neg)$ and $\neg \neg \neg A \rightarrow B \neg \neg \neg$, as $\text{Tr} (\neg \neg \neg A \rightarrow B \neg \neg \neg)$ is “local” whilst $\neg \neg \neg A \rightarrow B \neg \neg \neg$ is not.

In contrast, $\text{Tr} (\neg \neg \neg \text{Snow is white} \neg \neg \neg)$ and Snow is white are both “local”, which is why they can have the same subject matters, etc.
What about self-referential sentences in our HYPE-model for type-free truth?

(Consider the Liar $\lambda$: $\neg \text{Tr}(\neg \lambda \downarrow)$ and the Curry sentence $\chi$: $\text{Tr}(\neg \chi \downarrow) \rightarrow \bot$.)

- The least state $s_l$ neither satisfies the Liar $\lambda$ nor its negation $\neg \lambda$.
- The greatest state $s_g$ satisfies both the Liar $\lambda$ and its negation $\neg \lambda$.
- For all states $s$:
  
  $s \models_{\text{Tr}} \text{Tr}(\neg \lambda \downarrow) \leftrightarrow \lambda$ and $s \models_{\text{Tr}} \neg \text{Tr}(\neg \lambda \downarrow) \leftrightarrow \neg \lambda$,

- Every state $s$ satisfies the Curry sentence $\chi$ (in virtue of no state $s$ satisfying $\text{Tr}(\neg \chi \downarrow)$), and no state $s$ satisfies its negation $\neg \chi$.

- For all states $s$:

  $s \not\models_{\text{Tr}} (\text{Tr}(\neg \chi \downarrow) \leftrightarrow \chi)$ and $s \not\models_{\text{Tr}} (\neg \text{Tr}(\neg \chi \downarrow) \leftrightarrow \neg \chi)$,
  
  $s \models_{\text{Tr}} \neg (\text{Tr}(\neg \chi \downarrow) \leftrightarrow \chi)$ and $s \models_{\text{Tr}} \neg (\neg \text{Tr}(\neg \chi \downarrow) \leftrightarrow \neg \chi)$. 

Possible States Semantics for HYPE

Now we expand HYPE-models $\mathcal{M}$ to $p(ossible) s(states)-models \langle \mathcal{M}, R \rangle$:

- $R \subseteq S \times S$ is an “accessibility” relation, such that
  - if $s_1 \circ s_2$ is defined and $s_1 \circ s_2 R s'_2$, then there is an $s'_1$ with $s'_1 \leq s'_2$ and $s_1 R s'_1$;
  - if $s_1 \circ s_2$ is defined and $s_1 R s'_1$, then there is an $s'_2$ with $s'_1 \leq s'_2$ and $s_1 \circ s_2 R s'_2$;
  - [OPTIONAL:] if $s R s'$ then $s^* R s'^*$. 

- $s \models \Box A$ iff for all $s'$ with $s R s'$: $s' \models A$.

- $s \models \Diamond A$ iff there is an $s'$ with $s R s'$, such that: $s' \models A$.

Monotonicity and the Star Lemma follow as before. The duality of $\Box$ and $\Diamond$ follows if OPTIONAL is accepted.

If all states are worlds, a ps-model is simply a possible worlds model.
Modal HYPE system: HYPE +

- □(A → B) → (□A → □B).
- □(A ∧ B) ↔ (□A ∧ □B).
- □(A ∨ B) → (□A ∨ ◊B).
- □A ∧ ◊(A → B) → ◊B.
- ◊A ∧ □(A → B) → ◊B.
- □A ↔ ¬◊¬A.

MP, Substitution (substituting provably equivalent formulas for each other preserves theoremhood).

\[ \frac{\vdash A}{\vdash □A} \]

**Theorem**

*Modal HYPE is sound & complete with respect to ps-models with OPTIONAL.*

The methodology of Kripke semantics translates to modal HYPE: e.g., the transitivity of R yields the truth of □A → □□A at all states, etc.
Example: ps-models for the Seeing operator $S$ ("... sees that")

Accessible states may be thought to represent *images* compatible with what was seen.

$S$ satisfies the □-axioms and rules from before (except for OPTIONAL), with the addition of factivity: $S(A) \rightarrow A$.

$S$ creates a hyperintensional context, though the external logic in $w$ is classical.
Example: ps-models for *Determination D* and *Necessity □*

\[
\begin{align*}
    r &= (1 - p) \cdot q \\
    s &= \max(p, r)
\end{align*}
\]

\[
\begin{align*}
    p, q &\mapsto \bar{r}, \quad p, \bar{q} &\mapsto \bar{r}, \quad \bar{p}, q &\mapsto r, \quad \bar{p}, \bar{q} &\mapsto \bar{r} \\
    p, r &\mapsto s, \quad p, \bar{r} &\mapsto s, \quad \bar{p}, r &\mapsto s, \quad \bar{p}, \bar{r} &\mapsto \bar{s}
\end{align*}
\]

Expand the full HYPE-model to a ps-model (states \(\approx\) variable assignments):

- by defining a (functional) \(R\) through determination in a structural equations model, and by interpreting \(D\) (“it is determined that”) by means of \(R\) (e.g., \(p \xrightarrow{R} p\bar{r}s, q \xrightarrow{R} q, r \xrightarrow{R} rs, pq \xrightarrow{R} pq\bar{r}s, pqr \xrightarrow{R} pqr\bar{r}s, \ldots\));

- interpreting \(\Box\) (“it is necessary that”) by means of the total \(R'\) on \(S\).

In the example above, the following hold at all states:

- \(\Box(p \land q \rightarrow D(s))\).
- \(\Box(\neg p \land q \rightarrow D(s))\).
- \(\neg \Box(q \rightarrow D(s))\), even though all *worlds* that satisfy \(q\) satisfy \(D(s)\).
Example: ps-models for *Determination D* and *Necessity* □

Expand the full HYPE-model to a ps-model (states ≈ network states):

- by defining a (functional) $R$ through determination in a quasi-neural-network model, and by interpreting $D$ (“it is determined that”) by means of $R$ (e.g., $p \xrightarrow{R} p\overline{r}s$, $q \xrightarrow{R} q$, $r \xrightarrow{R} rs$, $pq \xrightarrow{R} pq\overline{r}s$, $pqr \xrightarrow{R} pqr\overline{r}s$, ...);
- interpreting □ (“it is necessary that”) by means of the total $R'$ on $S$.

In the example above, the following hold at all states:

- □($p \land q \rightarrow D(s)$).
- □($\neg p \land q \rightarrow D(s)$).
- $\neg □(q \rightarrow D(s))$, even though all *worlds* that satisfy $q$ satisfy $D(s)$. 
By combining the axioms for $D$ and $\Box$, one gets a nice axiomatization for necessary determination.

One of the axioms for $D$:

\[ \vdash (A \rightarrow D(A)) \land (D(A) \leftrightarrow D(D(A))) \]

For all words $w$ compatible with the equations/network:

\[ w \models A \leftrightarrow D(A). \]

Necessary determination is monotonic:

- If $s \models \Box(A \rightarrow D(B))$, then $s \models \Box(A \land C \rightarrow D(B))$.

That is one respect, next to others, in which the semantics of necessary determination differs from that of $because$.

Let us finally turn to ps-models for $because$. 
Possible States Models for *Because*

I take some inspiration from Mackie (1980) on *causation*:

Then in the case described above the complex formula ‘(*ABC* or *DGH* or *JKL*)’ represents a condition which is both necessary and sufficient for *P*: each conjunction, such as ‘*ABC*’, represents a condition which is sufficient but not necessary for *P*. Besides, *ABC* is a *minimal* sufficient condition: none of its conjuncts is redundant: no part of it, such as *AB*, is itself sufficient for *P*. But each single factor, such as *A*, is neither a necessary nor a sufficient condition for *P*. Yet it is clearly related to *P* in an important way: it is an *insufficient* but *non-redundant* part of an *unnecessary* but *sufficient* condition: it will be convenient to call this (using the first letters of the italicized words) an *inus* condition.⁵

*A, B, C,*… above can also be negative: e.g., *C* is the absence of *C*.

E.g.: “It may be the consumption of a certain poison conjoined with the non-consumption of the appropriate antidote which is invariably followed by death.”
Possible States Models for *Because*

I take some inspiration from Mackie (1980) on *causation*:

Then in the case described above the complex formula ‘(*ABC* or *DGH* or *JKL*)’ represents a condition which is both necessary and sufficient for *P*: each conjunction, such as ‘*ABC*’, represents a condition which is sufficient but not necessary for *P*. Besides, *ABC* is a *minimal* sufficient condition: none of its conjuncts is redundant: no part of it, such as *AB*, is itself sufficient for *P*. But each single factor, such as *A*, is neither a necessary nor a sufficient condition for *P*. Yet it is clearly related to *P* in an important way: it is an *insufficient* but *non-redundant* part of an *unnecessary* but *sufficient* condition: it will be convenient to call this (using the first letters of the italicized words) an *inus* condition.5

More than one notion of *cause* available:

Each of *A* vs *ABC* vs *ABC ∨ DGH ∨ JKL* may be said to be cause of *P*. 
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Then in the case described above the complex formula ‘\((ABC\) or \(DGH\) or \(JKL)\)’ represents a condition which is both necessary and sufficient for \(P\): each conjunction, such as ‘\(ABC\)’, represents a condition which is sufficient but not necessary for \(P\). Besides, \(ABC\) is a *minimal* sufficient condition: none of its conjuncts is redundant: no part of it, such as \(AB\), is itself sufficient for \(P\). But each single factor, such as \(A\), is neither a necessary nor a sufficient condition for \(P\). Yet it is clearly related to \(P\) in an important way: it is an *insufficient* but *non-redundant* part of an *unnecessary* but *sufficient* condition: it will be convenient to call this (using the first letters of the italicized words) an *insu* condition.\(^5\)

More than one notion of *cause* available:

Each of \(A\) vs \(\overline{ABC}\) vs \(\overline{ABC} \lor \overline{DGH} \lor \overline{JKL}\) may be said to be cause of \(P\).
I will now introduce a formal semantics for ‘because’ inspired by Mackie’s account of causation.

However, necessary determination in ps-models based on structural equations/networks does not need to be interpreted causally:

> there may be a distinctive kind of metaphysical explanation, in which explanans and explanandum are connected, not through some sort of causal mechanism, but through some constitutive form of determination (Fine 2012)

See Schaffer (2016) for arguments that grounding may be analyzed formally by means of structural equation models (“in the image of causation”).
Define: $s$ is a minimal $A$-state iff

$$s \models A, \text{ and there is no } s' \text{ with } s' < s \text{ and } s' \models A.$$  

Consider ps-models for $D$ and $\Box$ as before, but introduce an additional polyadic operator $\leq$ ("weakly-because"), such that:

$$s \models A_1, \ldots, A_n \leq B \text{ ("B weakly-because } A_1, \ldots, A_n\")} \text{ iff}$$

- (Factivity) $s \models A_1 \land \ldots \land A_n \land B$,
- (Sufficiency) $s \models \Box(A_1 \land \ldots \land A_n \rightarrow D(B))$,
- (Minimality) for all $s'$, such that $s'$ is a minimal $A_1 \land \ldots \land A_n$-state:
  
  there is no $s''$ with $s'' < s'$ and $s'' \models D(B)$.

  (That is: every minimal $A_1 \land \ldots \land A_n$-state is a minimal $D(B)$-state.)

(Syntactically, I use Fine’s 2012 format of “weak full grounding”.)
Example: the previous quasi-neural-network reconsidered

\[\begin{align*}
& (p \xrightarrow{R} p\overline{r}s, \quad q \xrightarrow{R} q, \quad r \xrightarrow{R} rs, \quad pq \xrightarrow{R} pq\overline{r}s, \quad pqr \xrightarrow{R} pqr\overline{r}s, \ldots); \\
\text{In the example above, we have:} \\
& \quad \bullet \ pq\overline{r}s \models p \leq s. \\
& \quad \bullet \ pq\overline{r}s \models \lnot(q \leq s) \land \lnot(r \leq s) \land \lnot((p \land q) \leq s) \land \lnot((p \land (q \lor \lnot q)) \leq s). \\
\text{Preemption cases are normally supposed to trouble Mackie’s account—not so in our hyperintensional setting! (Double prevention is unproblematic, too.)}
\end{align*}\]
As we have seen, satisfaction for $\leq$ quantifies over “minimal states”:

$s \models A_1, \ldots, A_n \leq B$ (“$B$ weakly-because $A_1, \ldots, A_n$”) iff

- $s \models A_1 \land \ldots \land A_n \land B$,
- $s \models □(A_1 \land \ldots \land A_n \to D(B))$,
- for all $s'$, such that $s'$ is a minimal $A_1 \land \ldots \land A_n$-state:
  
  there is no $s''$ with $s'' < s'$ and $s'' \models D(B)$.

That is unproblematic if one expands the full model to a ps-model for $D, □, \leq$.

But, more generally, one should actually use instead...
...this satisfaction clause, which does not quantify over minimal states:

\[ s \models A_1, \ldots, A_n \leq B \text{ ("B weakly-because } A_1, \ldots, A_n") \text{ iff} \]

\[- s \models A_1 \land \ldots \land A_n \land B, \]

\[- s \models \Box(A_1 \land \ldots \land A_n \rightarrow D(B)), \]

\[- \text{ for all nonempty } S' \subseteq S, \text{ such that} \]

(i) \( S' \) is totally ordered by \( \leq \),

(ii) for all \( s' \) in \( S' \), \( s' \models A_1 \land \ldots \land A_n \),

(iii) there is no \( s'' \), such that \( s'' < s' \) for all \( s' \) in \( S' \), and \( s'' \models A_1 \land \ldots \land A_n \): there is no \( s'' \), such that \( s'' < s' \) for all \( s' \) in \( S' \), and \( s'' \models D(B) \).

(In words: "no } D(B)\text{-state lies strictly below some } A_1 \land \ldots \land A_n\text{-exhausting descending chain".})

Compare: (No) Limit Assumption in Stalnaker/Lewis-semantics for } \Box\rightarrow.\]
Some logical validities and invalidities involving $\leq$:
(cf. Schnieder 2011, Fine 2012, . . .)

- $\vdash \top \leq \top$.
- $A_1, \ldots, A_n \leq B$, $\Box(A_1 \leftrightarrow A'_1), \ldots, \Box(A_n \leftrightarrow A'_n)$, $\Box(B \leftrightarrow B') \vdash A'_1, \ldots, A'_n \leq B'$.
- $A_1, \ldots, A_n \leq B \vdash A_1 \land \ldots \land A_n \land B$.
- $A_1, \ldots, A_n \leq B \vdash \Box(A_1 \land \ldots \land A_n \rightarrow D(B))$. 
Transitivity: $A \leq B, B \leq C \vdash A \leq C$.

$A \leq C, B \leq C \vdash (A \lor B) \leq C$.

$A \leq B, A \leq C \vdash A \leq (B \lor C)$.

$A \leq B, A \leq C \vdash A \leq (B \land C)$.

$\not\vdash A \to (A \leq A)$. (E.g., $pq\bar{r}s \not\models p \land s \to (p \land s \leq p \land s)$)

(But $A \to (A \leq A)$ is satisfied for all literals $A$ in any $\leq$-model that is based on the full model.)

$A \leq B, \Box (B \to C) \not\vdash A \leq C$. (E.g., $pq\bar{r}s \not\models p \leq \top$)

$A \leq C \not\vdash (A \land B) \leq C$. (Even Cautious Monotonicity is invalid.)

$A \land B \not\vdash (A, B \leq (A \land B))$. (But for all “initial” literals in the net: $\checkmark$)

$\not\vdash (A \to (A \leq (A \lor B)))) \land (B \to (B \leq (A \lor B))))$. (But for all literals: $\checkmark$)
Further notions of *because* can be defined in terms of *weakly-because* (cf. Fine 2012); for instance:

\[
A_1, \ldots, A_n < B \text{ ("B strictly-because } A_1, \ldots, A_n") \leftrightarrow_{df} \]

\[
(A_1, \ldots, A_n \leq B) \land \neg (B \leq A_1) \land \ldots \land \neg (B \leq A_n).
\]

< can then be shown to be irreflexive, asymmetric, transitive,…

The analysis can be extended to *partial* notions of *because*; and so forth.
Conclusions

- HYPE is a system of hyperintensional semantics and logic, which
  - is simple, formally nice, and concerns structures known from maths,
  - includes well-known systems as special cases or subsystems,
  - has nice applications, such as to the semantic paradoxes.

- HYPE may be extended to a possible states semantics for certain
  hyperintensional operators, such as
  - seeing, necessary determination, . . .
  - (a particular type of weak) because.

  (The next step is to study further notions of because in detail.)

- There are many open questions: e.g., what is the complete logic of \( \leq \)?

- There are many further applications: e.g., combine Leitgeb (2019) and the
  present account to give a joint theory of type-free truth and because